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Shock waves and non-stationary flow in a duct of varying cross-section

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Sound waves of finite but small amplitude propagating into a quasi-steady, supersonic flow in a non-uniform duct are analyzed by means of a perturbation method. General properties of the flow and of the wave propagation are studied using a one-dimensional approximation. A shock propagation law in the unsteady flow is obtained. As an example, the formation and development of shock waves are discussed for a duct with a conical convergence. Comparisons of the theory with an experiment are also made; fairly good agreement is found.

1. Introduction

Meyer (1951) investigated one-dimensional, unsteady flow of inviscid gas in a duct of slowly varying cross-section. For a small, but finite-amplitude, wave propagating into a steady flow, he found distributions of the flow velocity and pressure, the asymptotic behaviour of sound waves and the equation for the shock path. If a disturbance of finite extent and duration is set up in an originally steady, shock-free flow, it generates two primary waves, one advancing with velocity $u+a$ and the other receding with velocity $u-a$, where u and a are the velocities of the flow and the sound wave respectively. When the flow is supersonic, these two waves travel in the same direction, i.e. downstream, and, after a certain period of interaction, separate from each other to form two wave trains. If the duct has a throat where the steady flow is sonic, all receding wave fronts entering from upstream are decelerated to approach the sonic throat asymptotically, which leads to the formation of a shock. The analysis of wave propagation into steady flow is thus very much simplified if we are concerned with the propagation of one or another wave train after they separate.

However, conditions are different for wave propagation in unsteady flow, for which the duration of a disturbance cannot be restricted to a finite time. We must necessarily consider the two kinds of waves simultaneously. Moreover, for unsteady supersonic flow, deceleration of the flow may be caused not only by cross-sectional convergence of the duct but also by time variation of the flow quantities. If the flow velocity u^* at a given position decreases to the sound speed, the velocity of the receding wave vanishes somewhere in the convergence downstream. As u^* decreases further, the wave turns its direction to propagate upstream, leading to characteristic crossing and shock formation. The shock wave thus formed propagates upstream. The propagation is also affected continuously by the spatial inhomogeneity and the time variation of the flow.

Theoretical studies on shock wave propagation in a moving medium have been limited to the propagation into the steady flow. Apart from Meyer, Chester (1960) has studied shock waves of arbitrary strength using the rule proposed by Whitham (1958), and, for the upstream-facing weak shock, he obtained the relation

$$\delta(\delta-\lambda) = \text{const.},$$

where δ is the deviation of the shock Mach number from unity relative to the flow, and λ is that of the flow Mach number in front of the shock. Propagation into a steady uniform flow has been investigated also by Chisnell (1965) and Whitham (1968). However, effects of the time variation of the flow on the formation and propagation of shock waves were not discussed in detail by these authors.

The present paper deals with wave propagation in a quasi-steady flow in a duct of varying cross-section. The flow is assumed supersonic, but with velocity only slightly greater than the sound speed by the order of a smallness parameter ϵ. Then, the velocity of the advancing wave is of the order of $2a$, while that of the receding wave of the order of ϵ. In this case, by means of a perturbation method (Asano & Taniuti 1969, 1970), it is found that this difference of the velocities enables us to consider the problem in terms of one particular family of characteristics, namely those corresponding to receding waves.

One-dimensional motion of an inviscid, perfect gas is considered and the following conditions are assumed: (1) the amplitude of the wave is small but finite and (2) the change of cross-section of the duct is small. Physical quantities are expanded, about a constant state, as series in the small parameter ϵ, which is so chosen that the series for the cross-section is truncated, and the time variation of the flow is specified by the boundary condition on the flow at a position in the duct. Following Asano & Taniuti (1969, 1970) we use a stretched co-ordinate ϵt for time, which represents the slowness of the change of the physical quantities in time.

In § 2, a propagation law of the receding wave is obtained from the fundamental equations by the method of characteristics. In § 3, the law of shock-wave propagation is derived and compared with that given by Whitham's method. As an example, the case of a cylindrical duct with a conical convergence is studied in § 4. Conditions for shock formation in the system are also given. In § 5, the theory is compared with experimental results for a hydromagnetic plasma. The agreement is fairly good. Some related problems are briefly discussed in the last section, § 6.

2. Propagation of acoustic waves

We consider a flow passing through a duct of variable cross-section. In the one-dimensional approximation, the fluid motion is governed by the set of equations:

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}+\frac{1}{s}\frac{ds}{dx}\rho u = 0, \tag{2.1a}$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}=0, \tag{2.1b}$$

$$\frac{\partial w}{\partial t}+\frac{\partial q}{\partial x}+\frac{1}{s}\frac{ds}{dx}q=0, \tag{2.1c}$$

where the x axis is directed along the axis of the duct, whose cross-section is denoted by s. The notations are usual ones; that is, ρ, u and p are the mass density, the flow velocity and the pressure respectively, w and q are the energy density and its flux given by the equations

$$\left.\begin{aligned} w &= \tfrac{1}{2}\rho u^2+[1/(\gamma-1)]p, \\ q &= (w+p)u; \end{aligned}\right\} \tag{2.1d}$$

here γ is the adiabatic constant. The energy conservation equation (2.1c) is reduced to a simpler form by substituting (2.1a), (2.1b) and (2.1d) into (2.1c), i.e.

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\gamma p\frac{\partial u}{\partial x}+\frac{1}{s}\frac{ds}{dx}\gamma up=0. \tag{2.1c$'$}$$

Hence, we may take (2.1a), (2.1b) and (2.1c)$'$ as the fundamental equations.

Introducing a column vector U through the equation

$$U=\begin{bmatrix}\rho\\ u\\ p\end{bmatrix},$$

we write (2.1) in the matrix form

$$U_t+AU_x+(1/s)s_xB=0, \tag{2.2a}$$

where the subscripts t and x denote partial differentiation with respect to t and x respectively; A is a matrix of the form

$$A=\begin{bmatrix}u & \rho & 0\\ 0 & u & 1/\rho\\ 0 & \gamma p & u\end{bmatrix}, \tag{2.2b}$$

and B is the column vector

$$\begin{bmatrix}\rho u\\ 0\\ \gamma up\end{bmatrix}. \tag{2.2c}$$

If the duct is uniform, i.e. $s_x=0$, (2.2a) admits a solution with constant state, say U_0. Let U be expanded about U_0 as a series in a small parameter. The shock wave is supposed to be formed when the flow velocity becomes nearly equal to the sound velocity and we take U_0 as a state where u is equal to the sound velocity $a_0=(\gamma p_0/\rho_0)^{\frac{1}{2}}$, that is,

$$U_0=\begin{bmatrix}\rho_0\\ a_0\\ p_0\end{bmatrix}. \tag{2.3}$$

In the steady state, the deviation of U from U_0 essentially depends on the change of the cross-section. This can be seen most easily by the well-known relation (Liepmann & Roshko 1960)

$$s^2\propto(1/M^2)\{1+\tfrac{1}{2}(\gamma-1)M^2\}^{(\gamma+1)/(\gamma-1)},$$

where M is the Mach number of the flow. For the variation of Mach number when nearly equal to unity, the relation yields

$$dM \propto \frac{1}{M-1}\frac{ds}{s}. \tag{2.4}$$

This equation suggests that, if the variation of the cross-section is of the order of ϵ^2, the changes of the fluid quantities are of the order of ϵ for M nearly equal to 1. Hence, if the variation of the cross-section s is given by the equation

$$s = s_0 + \epsilon^2 s_1, \tag{2.5}$$

U is expanded as a series in powers of ϵ, that is,

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \dots. \tag{2.6a}$$

Other quantities that are functions of U can be expanded also as

$$\left.\begin{aligned} A &= A_0 + \epsilon A_1 + \dots, \\ B &= B_0 + \epsilon B_1 + \dots, \end{aligned}\right\} \tag{2.6b}$$

where A_0 and B_0 are the respective values of A and B at $U = U_0$.

As was stated in §1, we seek a solution to represent the receding waves, which travel with the velocity $u-a$. Since u_0 is put equal to a_0, the order of magnitude of the velocity $u-a$ is ϵ. Since $dx/dt = u-a \sim O(\epsilon)$, we introduce a new variable τ so as to give the relation $dx/dt \sim O(\epsilon)\, dx/d\tau$, where $dx/d\tau$ is of the order of unity, that is,

$$\tau = \epsilon t. \tag{2.7}$$

In other words, we consider a system which varies slowly in time, such that the characteristic time for a disturbance to propagate is $L/(u-a) \sim \epsilon^{-1}(L/a_0 M_1)$, where L is the characteristic length of the system and M_1, which is of the order of unity, is defined by the equation $M = 1 + \epsilon M_1 + \dots$.

Substituting (2.3)–(2.7) into (2.2a) and equating coefficients of equal power, we obtain for the first order of ϵ

$$A_0 U_{1x} = 0, \tag{2.8}$$

and for the second order

$$U_{1\tau} + A_0 U_{2x} + A_1 U_{1x} + \frac{1}{s_0} s_{1x} B_0 = 0, \tag{2.9}$$

and so on. From these equations, we can determine U_1 as follows.

In order that the non-trivial solution of (2.8) exists, it is required that $\det A_0 = 0$, which is satisfied automatically by virtue of the relation

$$u_0^2 = a_0^2 = \gamma p_0/\rho_0.$$

Let r be a right eigenvector of A_0 corresponding to the eigenvalue zero. Then, the general solution of (2.8) is given by

$$U_1 = r\bar{u}_1 + V(\tau), \tag{2.10}$$

where $\bar{u}_1$ is an arbitrary function of x and τ to be determined later, whence $V(\tau)$ is an arbitrary vector-valued function of τ only, that is,

$$V = \begin{bmatrix} v_1(\tau) \\ v_2(\tau) \\ v_3(\tau) \end{bmatrix},$$

and will be determined by boundary conditions on U_1 and $\bar{u}_1$. As the representation for r, we use, hereafter,

$$r = \begin{bmatrix} -\rho_0 \\ a_0 \\ -\gamma p_0 \end{bmatrix}.$$

The boundary condition on U_1 may be given by the value of U_1 at a point, say x_c, as a function of τ, i.e. $U_1(x_c, \tau)$. For the function $\bar{u}_1$, a convenient condition is to put $a_0 \bar{u}_1(x_c, \tau)$ equal to $u_1(x_c, \tau)$ which makes $v_2(\tau)$ vanish identically by means of the representation of r given above. Then, from (2.10) evaluated at $x = x_c$, v is given by

$$\left.\begin{aligned} v_1(\tau) &= \rho_0\{\bar{u}_1(x_c, \tau) + \bar{\rho}_1(x_c, \tau)\}, \\ v_2(\tau) &= 0, \\ v_3(\tau) &= \gamma p_0\{\bar{u}_1(x_c, \tau) + (1/\gamma)\,\bar{p}_1(x_c, \tau)\}, \end{aligned}\right\} \tag{2.11}$$

where the overbars denote dimensionless quantities; namely $\bar{\rho}_1(x, \tau)$, $\bar{u}_1(x, \tau)$ and $\bar{p}_1(x, \tau)$ are the density, the velocity and the pressure normalized by ρ_0, a_0 and p_0 respectively. Substituting (2.11) into (2.10) yields

$$\left.\begin{aligned} \bar{\rho}_1(x, \tau) &= \bar{\rho}_1(x_c, \tau) - \ \{\bar{u}_1(x, \tau) - \bar{u}_1(x_c, \tau)\}, \\ \bar{p}_1(x, \tau) &= \bar{p}_1(x_c, \tau) - \gamma\{\bar{u}_1(x, \tau) - \bar{u}_1(x_c, \tau)\}. \end{aligned}\right\} \tag{2.12}$$

For further discussion, it is convenient to express $\bar{\rho}_1$, $\bar{u}_1$ and $\bar{p}_1$ in terms of M_1. Expanding the sound velocity a as $a = a_0 + \epsilon a_1 + \ldots$ and noting the relation $\bar{a}_1 = a_1/a_0 = \frac{1}{2}(\bar{p}_1 - \bar{\rho}_1)$, we have, from (2.12),

$$\begin{aligned} M_1(x, \tau) &= \bar{u}_1 - \bar{a}_1 \\ &= \tfrac{1}{2}(\gamma+1)\{\bar{u}_1(x, \tau) - \bar{u}_1(x_c, \tau)\} + M_1(x_c, \tau), \end{aligned}$$

or

$$\bar{\rho}_1(x, \tau) = \bar{\rho}_1(x_c, \tau) - [2/(\gamma+1)]\{M_1(x, \tau) - M_1(x_c, \tau)\}, \tag{2.13a}$$

$$\bar{u}_1(x, \tau) = \bar{u}_1(x_c, \tau) + [2/(\gamma+1)]\{M_1(x, \tau) - M_1(x_c, \tau)\}, \tag{2.13b}$$

and

$$\bar{p}_1(x, \tau) = \bar{p}_1(x_c, \tau) - [2\gamma/(\gamma+1)]\{M_1(x, \tau) - M_1(x_c, \tau)\}. \tag{2.13c}$$

Hence our problem is reduced to one of obtaining $M_1(x, \tau)$.

Multiplying (2.9) by a left eigenvector l of A_0 corresponding to the eigenvalue zero, i.e. $l = (\frac{1}{2}a_0^2, \gamma a_0/(\gamma-1), -1)$ (Asano & Taniuti 1969, 1970), we have

$$lr\bar{u}_{1\tau} + lr\,.\,(\nabla_u A)_0 r\bar{u}_1\bar{u}_{1x} + lV\,.\,(\nabla_u A)_0 r\bar{u}_{1x} + lV_\tau + (1/s_0)\,s_{1x}lB_0 = 0, \tag{2.9$'$}$$

where we have used (2.10) and the relation

$$A_1 = U_1\,.\,(\nabla_u A)_0 = \rho_1(\partial A/\partial\rho)_0 + u_1(\partial A/\partial u)_0 + p_1(\partial A/\partial p)_0.$$

The equation for M_1 is easily obtained from (2.9)$'$ and (2.13) as

$$M_1 + a_0 M_1 M_{1x} - F(\tau) - \tfrac{1}{4}(\gamma+1)\,a_0(s_{1x}/s_0) = 0, \tag{2.14a}$$

where

$$F(\tau) = M_{1\tau}(x_c,\tau) + a_0 M_1(x_c,\tau)\, M_{1x}(x_c,\tau) - \tfrac{1}{4}(\gamma+1)\, a_0\,(s_{1x}(x_c)/s_0)$$
$$= M_{1\tau}(x_c,\tau) - \tfrac{1}{4}(\gamma+1)\,\{\bar{u}_{1\tau}(x_c,\tau) - (1/\gamma)\,\bar{p}_{1\tau}(x_c,\tau)\}$$
$$= \tfrac{1}{2}\{\bar{\rho}_{1\tau}(x_c,\tau) - \tfrac{1}{2}(\gamma-3)\,\bar{u}_{1\tau}(x_c,\tau) - [(\gamma-1)/2\gamma]\,\bar{p}_{1\tau}(x_c,\tau)\}. \qquad (2.14b)$$

The equations (2.14) can be solved by means of the method of characteristics; on each characteristic curve

$$dx/d\tau = a_0 M_1, \qquad (2.15a)$$

the variation of M_1 is given by

$$dM_1 = F(\tau)\, d\tau + \tfrac{1}{4}(\gamma+1)\,(a_0/s_0)\,(ds_1/dx)\, d\tau$$
$$= F(\tau)\, d\tau + [(\gamma+1)/4M_1]\, ds_1/s_0, \qquad (2.15b)$$

where ds_1 is the change of the cross-section experienced by the sound wave, i.e. $ds_1 = (ds_1/dx)\, a_0 M_1\, d\tau$. The set of equations (2.15) is an extension of the relation (2.4) to the non-stationary case for the small variation of M from unity and is integrated to give

$$M_1(\tau) = M_1(\tau_c) + \int_{\tau_c}^{\tau} F(\tau)\, d\tau + \tfrac{1}{4}(\gamma+1)\, a_0 \frac{1}{s_0}\int_{\tau_c}^{\tau} \frac{ds_1}{dx}\, d\tau, \qquad (2.16a)$$

$$x = x_c + a_0 \int_{\tau_c}^{\tau} M_1(\tau)\, d\tau, \qquad (2.16b)$$

where τ_c is the time at which each characteristic curve issues out of the position $x = x_c$.

For steady supersonic flow, the relation (2.4) implies that the flow is accelerated in a diverging duct for which $ds/dx > 0$. However, for the flow whose Mach number decreases at the place $x = x_c$, i.e. $F(\tau) < 0$, equation (2.15b) shows that the flow may be decelerated even if ds/dx is positive; thus, the boundary conditions play an important role in shock formation, as will be shown in the following sections.

3. Propagation of shock waves

The characteristics determined by (2.15) may cross each other after finite time to form a discontinuity, which propagates as a shock wave under certain conditions. The shock propagation in an inhomogeneous medium is governed by Whitham's rule which comprises the shock condition and the characteristic relation. However, as far as the propagation of weak shocks is concerned, one may work on the weak extension of (2.14). Since (2.14a) can be written in the form of a conservation law, the jump condition for weak shocks is obtained by applying Gauss's theorem to that equation (Jeffrey & Taniuti 1964) as

$$2M_s = M_{1f} + M_{1b}, \qquad (3.1)$$

along the shock trajectory. Here, subscripts f and b denote the quantities just in front of and behind the shock respectively and M_s is the shock Mach number defined by

$$dx/d\tau = a_0 M_s. \qquad (3.2)$$

It is to be noted that the relation (3.1) is an approximation of the exact jump condition. We now show how the shock trajectory can be determined from (3.2).

Eliminating τ_c from (2.16) yields the Mach number

$$M_1 = M_1(x, \tau) \tag{3.3}$$

for any x and τ except those on the shock trajectory, and hence the Mach numbers M_{1f} and M_{1b} are obtained from (3.3) evaluated just in front of and behind the shock trajectory (3.2). Thus substituting (3.1) and (3.3) into (3.2), we have the equation

$$dx/d\tau = \tfrac{1}{2}a_0\{M_{1f}(x,\tau) + M_{1b}(x,\tau)\},$$

which gives the shock trajectory if it is solved for x in terms of τ.

For discussion of the properties of the shock wave, however, it may be more convenient to give the variation of the shock Mach number along the shock trajectory (3.2). To this end, let us divide the change of M_s along the shock trajectory into that along the characteristics (2.15a) and that away from the characteristics. This can be achieved easily by means of the transformation of the independent variables from x, τ to τ, τ_c, and is easily carried out by the Jacobian formulae,

$$\left(\frac{\partial \phi}{\partial x}\right)_y = \frac{\partial(\phi, y)}{\partial(x, y)}, \quad \frac{\partial(\phi, \psi)}{\partial(x, y)} = \frac{\partial(\phi, \psi)}{\partial(s, t)}\,\frac{\partial(s, t)}{\partial(x, y)}, \quad \text{etc.}$$

Using these formulae, we have

$$\left(\frac{\partial M_1}{\partial \tau}\right)_x = \left(\frac{\partial M_1}{\partial \tau}\right)_{\tau_c} - a_0 M_1(\tau, \tau_c)\left[\left(\frac{\partial M_1}{\partial \tau_c}\right)_\tau \Big/ \left(\frac{\partial x}{\partial \tau_c}\right)_\tau\right] \tag{3.4a}$$

and

$$\left(\frac{\partial M_1}{\partial x}\right)_\tau = \left(\frac{\partial M_1}{\partial \tau_c}\right)_\tau \Big/ \left(\frac{\partial x}{\partial \tau_c}\right)_\tau. \tag{3.4b}$$

Hence the variation of the Mach number along the shock trajectory, ΔM_1, is given by

$$\begin{aligned}\Delta M_1 &= \left(\frac{\partial M_1}{\partial \tau}\right)_x \Delta\tau + \left(\frac{\partial M_1}{\partial x}\right)_\tau \Delta x \\ &= \left\{\left(\frac{\partial M_1}{\partial \tau}\right)_x + a_0 M_s \left(\frac{\partial M_1}{\partial x}\right)_\tau\right\}\Delta\tau \\ &= \left[\left(\frac{\partial M_1}{\partial \tau}\right)_{\tau_c} + a_0\{M_s - M_1(\tau, \tau_c)\}\left[\left(\frac{\partial M_1}{\partial \tau_c}\right)_\tau \Big/ \left(\frac{\partial x}{\partial \tau_c}\right)_\tau\right]\right]\Delta\tau,\end{aligned} \tag{3.5}$$

where Δx and $\Delta\tau$ denote the respective variations of x and τ along the shock trajectory

$$\Delta x/\Delta\tau = a_0 M_s. \tag{3.6}$$

It is obvious that the measure of $\Delta\tau$ can be chosen arbitrarily. The first term $(\partial M_1/\partial\tau)_{\tau_c}$ in the bracket on the right-hand side of (3.5) is the variation of M_1 along the characteristics whilst the next term represents the variation in a direction away from the characteristics. Since the shock condition (3.1) gives

$$2\Delta M_s = \Delta M_{1f} + \Delta M_{1b}, \tag{3.7}$$

substituting (3.5) into (3.7) yields the desired decomposition of ΔM_s:

$$\Delta M_s = \frac{1}{2}\left[\left(\frac{\partial M_{1f}}{\partial \tau}\right)_{\tau_c} + \left(\frac{\partial M_{1b}}{\partial \tau}\right)_{\tau_c} + a_0(M_s - M_{1f})\left(\frac{\partial M_{1f}}{\partial x_f}\right)_\tau + a_0(M_s - M_{1b})\left(\frac{\partial M_{1b}}{\partial x_b}\right)_\tau\right]\Delta\tau$$

$$= \frac{1}{2}\left[\left(\frac{\partial M_{1f}}{\partial \tau}\right)_{\tau_c} + \left(\frac{\partial M_{1b}}{\partial \tau}\right)_{\tau_c} + a_0(M_s - M_{1f})\left\{\left(\frac{\partial M_{1f}}{\partial x_f}\right)_\tau - \left(\frac{\partial M_{1b}}{\partial x_b}\right)_\tau\right\}\right]\Delta\tau$$

$$= \frac{1}{2}\left[\left(\frac{\partial M_{1f}}{\partial \tau}\right)_{\tau_c} + \left(\frac{\partial M_{1b}}{\partial \tau}\right)_{\tau_c} + a_0(M_s - M_{1f})\left\{\left[\left(\frac{\partial M_{1f}}{\partial \tau_c}\right)_\tau \Big/ \left(\frac{\partial x_f}{\partial \tau_c}\right)_\tau\right] - \left[\left(\frac{\partial M_{1b}}{\partial \tau_c}\right)_\tau \Big/ \left(\frac{\partial x_b}{\partial \tau_c}\right)_\tau\right]\right\}\right]\Delta\tau. \tag{3.8}$$

Since further discussions of (3.8) in general form seem to be complicated, we study, as an example, the shock propagation in a steady flow.

The conditions of the steady flow are represented from (2.14*b*) and (3.4) as

$$F(\tau) = 0$$

and

$$\left(\frac{\partial M_1}{\partial \tau}\right)_x = \left(\frac{\partial M_1}{\partial \tau}\right)_{\tau_c} - a_0 M_1(\tau, \tau_c)\left(\frac{\partial M_1}{\partial x}\right)_\tau = 0.$$

Then, (3.5) and (2.15*b*) yield

$$\Delta M_1 = \frac{M_s}{M_1}\left(\frac{\partial M_1}{\partial \tau}\right)_{\tau_c}\Delta\tau = \frac{\gamma+1}{4}\frac{a_0}{s_0}\frac{M_s}{M_1}\frac{ds_1}{dx}\Delta\tau = \frac{\gamma+1}{4}\frac{1}{M_1}\frac{\Delta s_1}{s_0}. \tag{3.9}$$

Hence, substituting (3.9) into (3.7) gives

$$\Delta M_s = \frac{\gamma+1}{8}\left(\frac{1}{M_{1f}} + \frac{1}{M_{1b}}\right)\frac{\Delta s_1}{s_0},$$

which, by virtue of (3.1) and (3.9), can be integrated to give

$$M_s(M_s - M_{1f}) = \text{const.} \tag{3.10}$$

It is easy to see that the relation (3.10) reduces to Chester's (1960) one, if use is made of the Mach number relative to the flow in front. Following the original form of Whitham's rule, Chester applied the differential relations along the characteristics $dx/dt = u + a$ to the flow quantities just behind the shock wave. Since we consider only the characteristics $dx/dt = u - a$, there is no exact correspondence to Chester's derivation. The last equation in (3.9), however, shows that the variation of M_1 along the shock path has the same form as that along the characteristics (2.15*b*), being essentially the same as the result expected from Whitham's rule.

4. Shock formation in an axisymmetric duct with a convergence

In this section, using the general results obtained so far, we discuss the shock formation in an axisymmetric duct with a conical convergence and illustrate how the boundary condition works for the formation and propagation of the shock wave in the system.

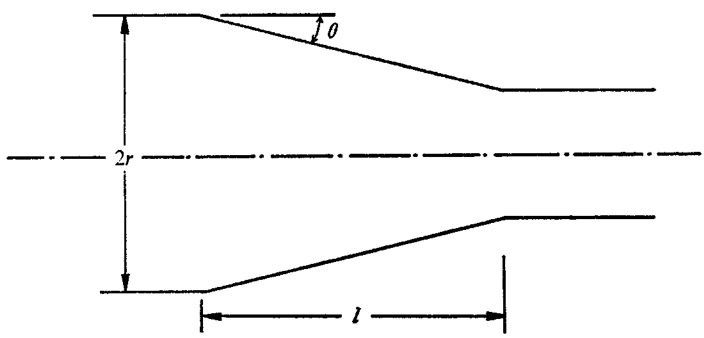

FIGURE 1. The axisymmetric duct with a conical convergence used in the analysis.

Let the radius of the cylindrical part of the duct be r, the open angle and the length of the cone be 2θ and l respectively, as shown in figure 1, and θ is assumed to be small. The x axis is directed along the axis of the cylinder, with the origin at the entrance of the convergence. Then, the cross-section s is given by

$$s = \begin{cases} s_0 \equiv \pi r^2 & (x \leqslant 0), \\ s_0(1-\theta(x/r))^2 & (0 \leqslant x \leqslant l), \\ s_l \equiv s_0(1-\theta(l/r))^2 & (x \geqslant l). \end{cases}$$

The second equation becomes, to the first order in θ,

$$s = s_0 - 2\pi r\theta x = s_0 + \epsilon^2 s_1(x),$$

where ϵ is defined by $\epsilon = \theta^{\frac{1}{2}}$ to give

$$s_1 = -2\pi r x. \tag{4.1}$$

For simplicity the boundary condition is given at the entrance of the convergence such that the flow velocity, the density and the pressure are decreasing linearly in time:

$$\left.\begin{aligned} \bar{\rho}_1(0,\tau) &= \bar{\rho}_1^0 - m_\rho \tau, \\ \bar{u}_1(0,\tau) &= \bar{u}_1^0 - m_u \tau, \\ \bar{p}_1(0,\tau) &= \bar{p}_1^0 - m_p \tau, \end{aligned}\right\} \tag{4.2}$$

where $\bar{\rho}_1^0$, $\bar{u}_1^0$, $\bar{p}_1^0$ and m's are positive constants. The Mach number M_1 at the entrance becomes

$$M_1(0,\tau) = m^0 - m'\tau, \tag{4.3a}$$

here m_0 is the Mach number at $\tau = 0$ and given as

$$m^0 = \bar{u}_1^0 - \tfrac{1}{2}(\bar{p}_1^0 - \bar{\rho}_1^0) \tag{4.3b}$$

and

$$m' = m_u - \tfrac{1}{2}(m_p - m_\rho). \tag{4.3c}$$

In general, m' may be negative, that is, the Mach number may increase even if the flow velocity decreases.

Substituting (4.1)–(4.3) into (2.14b) and (2.15b) with $x_c = 0$, we have, along the characteristics (2.15a),

$$dM_1 = n\,d\tau$$

or

$$M_1 = n(\tau - \tau_c) + M_{1c}, \tag{4.4}$$

where n is a constant which, in the region $0 \leqslant x \leqslant l$, takes the value

$$n = -m' + \frac{\gamma+1}{4}\left(m_u - \frac{1}{\gamma} m_p\right) - \frac{\gamma+1}{2r} a_0 \tag{4.5}$$

and M_{1c} is the Mach number at $\tau = \tau_c$

$$M_{1c} = m_0 - m'\tau_c.$$

Since we consider the case of decelerating flow, we assume n to be negative. By virtue of (4.4), (2.15a) is simply integrated to yield

$$\begin{aligned} x &= \tfrac{1}{2}a_0 n(\tau-\tau_c)^2 + a_0 M_{1c}(\tau-\tau_c) \\ &= a_0(\tau-\tau_c)(m_0 - \tfrac{1}{2}n'\tau_c + \tfrac{1}{2}n\tau), \end{aligned}$$

or

$$x = (a_0/2n)(M_1^2 - M_{1c}^2), \tag{4.6}$$

where n' is a constant defined by

$$\begin{aligned} n' &= n + 2m' \\ &= m' + \frac{\gamma+1}{4}\left(m_u - \frac{1}{\gamma} m_p\right) - \frac{\gamma+1}{2r} a_0. \end{aligned} \tag{4.7}$$

In the present case, the characteristics are parabolic curves with τ_c as a parameter to specify each curve. The starting-point of an envelope of these curves, if it exists, represents the formation of a shock wave. The initial shock velocity is approximately equal to M_1 at this point because, in the neighbourhood of this point, M_{1f} is almost equal to M_{1b}, and hence to M_s, as will be shown later.

It is easy to see that the curves (4.6) form an envelope

$$x = -(a_0/2n')(m^0 - m'\tau)^2. \tag{4.8}$$

Hence the necessary condition for shock formation in the convergence is

$$n' < 0. \tag{4.9}$$

The physical meaning of this condition is seen from the definition of n', (4.7); since n' is rewritten as

$$n' = dM_1/d\tau - 2(\partial M_1/\partial\tau)_{x=0},$$

the condition (4.9) is equivalent to

$$\frac{dM_1}{d\tau} - \left(\frac{\partial M_1}{\partial \tau}\right)_{x=0} < \left(\frac{\partial M_1}{\partial \tau}\right)_{x=0}.$$

The quantity on the left-hand side of this inequality may be interpreted as the change of the Mach number due to the convergence of the duct. Hence the inequality means that, if m' is positive so that M_1 is decreasing at the entrance, the rate of decrease of the flow Mach number due to the convergence is larger than that at the entrance. Thus, rapid decrease of the flow velocity at the entrance will delay the shock formation downstream.

Hereafter we consider only the case $n' < 0$ and $m' > 0$ – the flow Mach number decreases at the entrance of the convergence. Since n is also negative, characteristics (4.6) imply that there is a critical characteristic curve which is tangential to the line $x = l$, the exit of the convergence. This critical characteristic curve is specified by the parameter $\tau_c = \tau'_c$,

$$\tau'_c = (1/m')\,(m^0 - 2(|n|\,l/a_0)^{\frac{1}{2}}), \tag{4.10a}$$

and the corresponding Mach number at the entrance is

$$\begin{aligned} M'_{1c} &= m^0 - m'\tau'_c \\ &= 2(|n|\,la_0)^{\frac{1}{2}}. \end{aligned} \tag{4.10b}$$

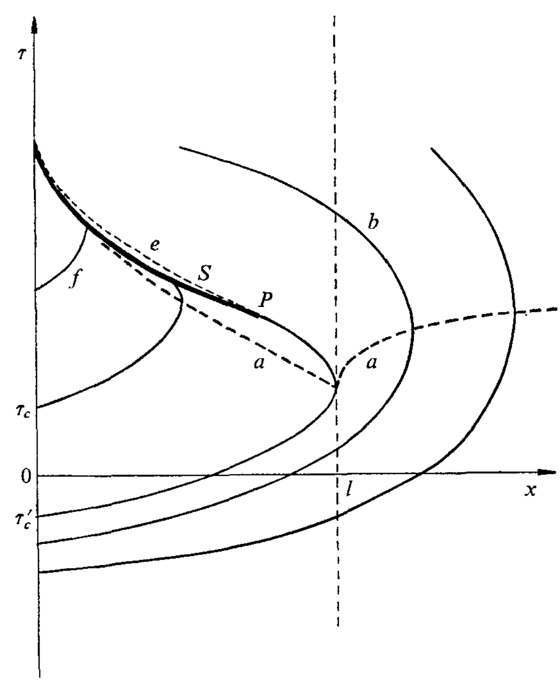

FIGURE 2. Flow diagram showing the formation and the motion of the shock wave. The shock wave is formed at P and propagates along the curve S. f and b denote the characteristics in front of and behind the shock respectively. The envelope of the characteristics for $\tau_c \geqslant \tau'_c$ is represented by the dotted curve e if the shock trajectory s is absent. The broken curve a denotes the sonic line.

When the flow Mach number at the entrance is large enough, disturbances to the flow pass through the convergence and a super–super transition is realized, but, as the Mach number decreases to M'_{1c}, the deceleration due to the convergence becomes effective and waves cannot pass through, leading to characteristic crossing. The existence of such a critical Mach number has been shown by Hamada, Kawakami & Sato (1968), though the value obtained differs a little from ours. The characteristic crossing occurs at a tangent point of the envelope (4.8) and the critical characteristic. In figure 2, the tangent point is denoted by P, the co-ordinates of which are given by

$$x_p = 4n'n/(n'+n)^2, \tag{4.11a}$$

$$\tau_p = 2(n'\tau_c'-m^0)/(n'+n). \tag{4.11b}$$

It is to be noted that x_p tends to l as $m' \to 0$; that is, the position of the shock formation tends to the exit of the duct as the flow becomes steady. Since the shock wave is formed at $x = x_p$ and $t = t_p$ and propagates upstream, characteristics with $\tau_c > \tau_c'$ and $\tau_c < \tau_c'$ become, after the time t_p, those in front of and behind the shock respectively. Thus, the characteristics (4.6) and the Mach number (4.4) represent x_f and M_{1f} respectively, while x_b and M_{1b} are obtained from further integration of (2.15) into the region $x \geqslant l$ and successively into the region $0 \leqslant x \leqslant l$, as shown in figure 2.

In order to determine the shock trajectory by means of the relation (3.1), the distributions of M_1 in space-time must be known. The flow Mach number in front of the shock is easily obtained by eliminating τ_c from (4.4) and (4.6):

$$M_{1f} = \left[1\Big/\frac{1}{m'}\left(\frac{1}{m'}+\frac{2}{n}\right)\right]\left\{\frac{1}{n}\left(\frac{m^0}{m'}-\tau\right)+\left(\frac{1}{m'}+\frac{1}{n}\right)\left[\left(\frac{m^0}{m'}-\tau\right)^2+\frac{1}{m'}\left(\frac{1}{m'}+\frac{2}{n}\right)\frac{2nx}{a_0}\right]^{\frac{1}{2}}\right\}. \tag{4.12}$$

After some calculations, we find the characteristics and Mach number behind the shock given by

$$x = (a_0/2n)(M_1^2-M_{1c}^2), \tag{4.13a}$$

and

$$M_1 = n(\tau-\tau_c)+M_{1c}+2[(n/\bar{n})-1][M_{1c}^2+(2nl/a_0)]^{\frac{1}{2}}, \tag{4.13b}$$

respectively, where $\bar{n}$ is a constant corresponding to n for $x \geqslant l$:

$$\begin{aligned}\bar{n} &= \lim_{\tau\to\infty} n \\ &= -m'+\tfrac{1}{4}(\gamma+1)(m_u-(1/\gamma)m_p).\end{aligned}$$

The characteristics that first form the discontinuity are specified by $\tau_c \cong \tau_c'$ or $M_{1c} \cong M_{1c}' = [2|n|l/a_0]^{\frac{1}{2}}$, hence, by virtue of (3.1), (4.13), (4.4) and (4.6) yield $M_{1f} \cong M_{1b} \cong M_s$; that is to say, the shock trajectory is represented by the characteristics with $\tau_c \cong \tau_c'$. On the other hand, for $\tau_c \ll \tau_c'$, we have

$$M_{1c} \gg [2|n|l/a_0]^{\frac{1}{2}}.$$

Consequently the distribution $M_{1b}(x,\tau)$ given by (4.13) takes the form

$$\begin{aligned}M_{1b} = \left[1\Big/\left(\frac{1}{m'}+\delta\right)\left(\frac{1}{m'}+\frac{2}{n}+\delta\right)\right]&\left\{\frac{1}{n}\left(\frac{m^0}{m'}-\tau\right)\right.\\ &\left.+\left(\frac{1}{m}+\frac{1}{n}+\delta\right)\left[\left(\frac{m^0}{m'}-\tau\right)^2+\left(\frac{1}{m'}+\delta\right)\left(\frac{1}{m'}+\frac{2}{n}+\delta\right)\frac{2nx}{a_0}\right]^{\frac{1}{2}}\right\},\end{aligned} \tag{4.14}$$

where we put

$$\delta = 2[(n/\bar{n})-1].$$

By means of (3.1), (4.12) and (4.14), (3.8) is easily integrated to give the shock trajectory for $\tau \gg \tau_p$,

$$x = K(m^0 - m'\tau)^2, \tag{4.15a}$$

where K is a constant determined by the equation

$$-\frac{4}{a_0}m'^2K = \left[1\Big/\frac{1}{m'}\left(\frac{1}{m'}+\frac{2}{n}\right)\right]\left\{\frac{1}{n}+\left(\frac{1}{m'}+\frac{1}{n}\right)\left[1+\frac{1}{m'}\left(\frac{1}{m'}+\frac{2}{n}\right)\frac{2nm'^2}{a_0}K\right]^{\frac{1}{2}}\right\}$$
$$+\left[1\Big/\left(\frac{1}{m'}+\delta\right)\left(\frac{1}{m'}+\frac{2}{n}+\delta\right)\right]\left\{\frac{1}{n}+\left(\frac{1}{m'}+\frac{1}{n}+\delta\right)\right.$$
$$\left.\times\left[1+\left(\frac{1}{m'}+\delta\right)\left(\frac{1}{m'}+\frac{2}{n}+\delta\right)\frac{2nm'^2}{a_0}K\right]^{\frac{1}{2}}\right\}. \tag{4.15b}$$

Comparing (4.15) with (4.8), we find that the shock trajectory is close to the envelope when the shock is weak. In figure 2, the shock trajectory is represented by a bold line. Broken lines in the figure show the forward and backward sonic lines, the former obtained from (4.12) with the condition $M_{1f} = 0$, i.e.

$$x = -\frac{n}{2(n+m)^2}a_0(m^0 - m'\tau)^2,$$

while the latter takes the form for $x \gtrsim l$

$$x = -\frac{\delta+2}{(m'\delta)^2}a_0\left[m^0 - \frac{n+m'}{n}\left(\frac{2|n|l}{a_0}\right)^{\frac{1}{2}} - m'\tau\right]^2 + l.$$

So far we have considered that the Mach number at the entrance of the convergence decreases from a sufficiently large value in the remote past. The Mach number at the entrance may, of course, be specified as constant for $\tau_c < \tau_c'$ provided that it is larger than M_{1c}. If it is a constant smaller than M_{1c}, the initial-boundary-value problem under consideration is not well posed, because, in this case, shock waves are necessarily formed in the remote past and propagate upstream.

5. Comparison with experiment

The present work was motivated by the B.S.G. project (Uchida *et al.* 1965, 1968, 1969) for plasma heating, at the Institute of Plasma Physics, Nagoya University. Conceptional profiles of magnetic lines of force in the system are depicted in figure 3. Ionized gas is produced by a theta pinch located at B and expands into S. In order to decelerate the plasma flow a magnetic mirror is set up at the centre or the end of S. Following the reference cited, we call these positions of the magnetic mirror S V and S VIII respectively. Under the experimental conditions, the plasma can be considered collision dominated. Furthermore the magnetic pressure is much higher than the mechanical pressure so that the magnetic field is not appreciably disturbed by the injection of the plasma. Hence the time variation of the magnetic field may be neglected; the lines of force act as a solid wall. A phenomenological theory of this experiment was established by Leloup & Taussig (1968). They showed that the essentially non-stationary stage, where the flow expands into S and its state changes swiftly in time, is followed by a quasi-steady stage for which the flow states at the entrance of the mirror field can be considered as slowly varying in time. From the experimental data, one sees that the boundary condition may be represented by such linear functions of time as equations (4.2). These results enable us to apply our theory to investigate this experiment.

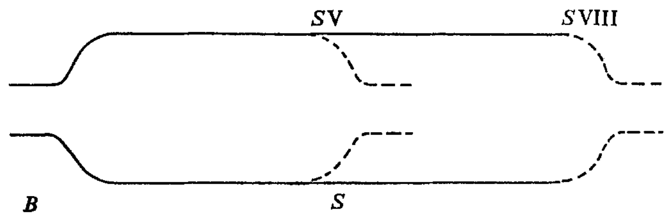

FIGURE 3. Sketch of the B.S.G. device. The curves represent the magnetic lines of force. The magnetic mirror is set up at SV or SVIII, as depicted by dotted lines.

We compare our theory with the experiment in respect of (1) the shock formation time t_p, (2) the initial shock velocity V_i and (3) the critical flow Mach number for the shock formation M_c'. The experimental data give these quantities as functions of the mirror ratio, the ratio of the magnetic field strength at the neck of the mirror to that at the homogeneous part S. Hence, for the comparison, the relation between the mirror ratio, say R, and the parameter ϵ is required. The angle θ is defined from the profile of the line of force. Noting that the mirror ratio was changed while keeping the S field constant, we have

$$\begin{aligned}\epsilon &= \theta^{\frac{1}{2}}\\ &= (r/l)^{\frac{1}{2}}(1-R^{-\frac{1}{2}})^{\frac{1}{2}},\end{aligned} \tag{5.1}$$

where we have used the conservation law of the magnetic flux and approximated the mirror field by a cone of the form shown in figure 1. From the design of the experimental device, we find $r/l \lesssim 0{\cdot}3$ which yields $\epsilon^2 \lesssim 0{\cdot}14$ for $R \leqslant 6$, and hence the theory can be applicable.

(1) From (2.7), (4.11*b*) and (5.1), we have

$$\begin{aligned}t_p &= \epsilon^{-1}\tau_p\\ &= K_1(1-R^{-\frac{1}{2}})^{-\frac{1}{2}}.\end{aligned} \tag{5.2}$$

Here we note that K_1 is a constant independent of R. The shock formation times at S V and S VIII are shown in figure 4. The origin of the time axis is adjusted so that the shocks at S V and S VIII are formed at the same time for $R = 5{\cdot}8$. The theoretical curve is also adjusted to fit the data at the same R.

(2) Since the initial shock velocities are not observed experimentally we identify the constant shock velocity measured at S with the initial shock velocity. The values of the initial shock velocities thus determined are shown in figure 5. On the other hand, the present theory yields the initial shock velocity V_i in the laboratory system, i.e. in the $x-t$ plane, as

$$\begin{aligned}V_i &= a\tilde{M}_{si} = \epsilon a_0 M_{si}\\ &= a_0(r/l)^{\frac{1}{2}}M_1(\tau_p, \tau_c')\,(1-R^{-\frac{1}{2}})^{\frac{1}{2}},\end{aligned}$$

where $\tilde{M}_{si}$ is the initial shock Mach number in the laboratory system. By virtue of the relation

$$M_1(\tau_p, \tau_c') = n(\tau_p - \tau_c') + M_{1c}'$$
$$= \frac{m'}{n+m'}\left(\frac{2|n|l}{a_0}\right)^{\frac{1}{2}},$$

the velocity V_i can be rewritten as

$$V_i = K_2 a_0^{\frac{1}{2}} (1 - R^{-\frac{1}{2}})^{\frac{1}{2}}, \tag{5.3}$$

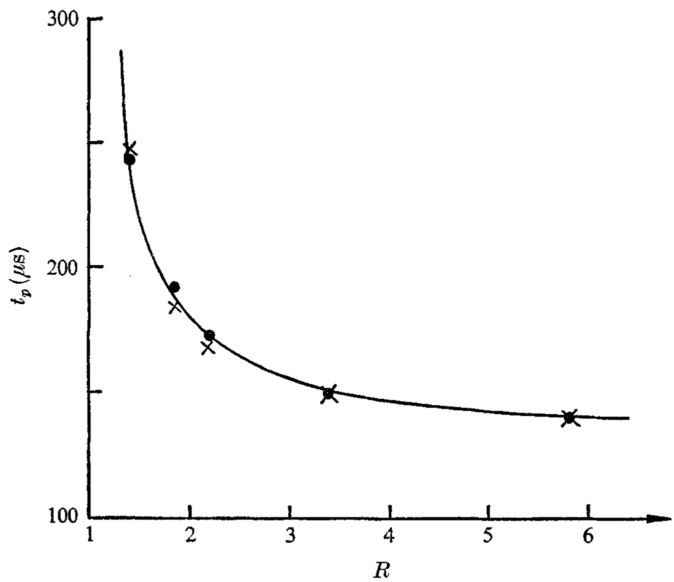

FIGURE 4. Shock formation time as a function of the mirror ratio. Theory gives the curve, which is fitted to the data at R = 5·8. ×, S V; ●, S VIII.

where K_2 is a constant. Since the times at which the plasma reaches S V and S VIII are different from each other, the temperatures and hence the sound velocities at S V and S VIII are different at the time of shock formation. Hence the difference of the initial shock velocities at S V and S VIII is due to that of the boundary condition. Let the sound velocities at S V and at S VIII be $a_{0\mathrm{V}}$ and $a_{0\mathrm{VIII}}$ respectively. Then from (5.3) the ratio of these two sound velocities is given by that of V_i^2, which, by means of figure 5, can be estimated as

$$\frac{a_{0\mathrm{V}}}{a_{0\mathrm{VIII}}} = 1{\cdot}55.$$

Consequently, the corresponding ratio of the temperature is about 2·40. The theoretical curves are fitted to the data at R = 5·8 in figure 5.

(3) By virtue of (4.10b) and (5.1), the critical Mach number $\tilde{M}_c'$ of the flow takes the form

$$\tilde{M}_c' = 1 + \epsilon M_{1c}'$$
$$= 1 + K_3 a_0^{-\frac{1}{2}} (1 - R^{-\frac{1}{2}})^{\frac{1}{2}},$$

where K_3 is another constant. Observed values and the theoretical curves are presented in figure 6, in which the constant $K_3/a_{0V}^{\frac{1}{2}}$ is adjusted to give the observed Mach number at $R = 5{\cdot}8$ and use is made of the ratio a_{0V}/a_{0VIII} given above. Since the present theory is valid for $\tilde{M}'_c \lesssim 2$, the discrepancy for $\tilde{M}'_c > 2{\cdot}5$ is an expected one. Thus the agreement of the theory with the experiment is fairly good within the approximation.

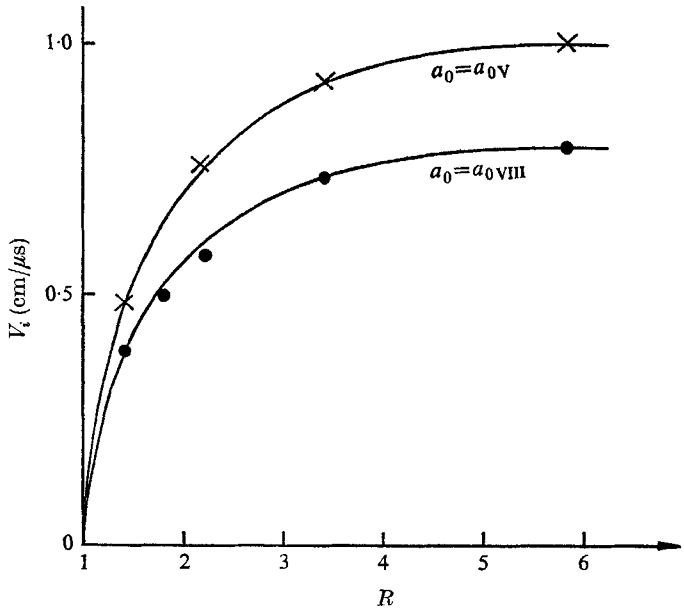

FIGURE 5. Initial shock velocities as functions of the mirror ratio. Theory gives the curves, which are fitted to the data at $R = 5{\cdot}8$. ×, S V; ●, S VIII.

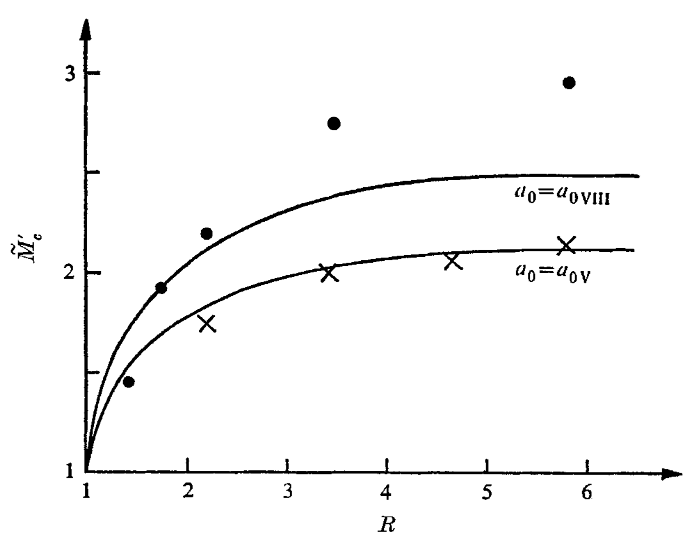

FIGURE 6. Critical flow Mach numbers as functions of the mirror ratio. The curve for $a_0 = a_{0V}$ is fitted to the data at $R = 5{\cdot}8$. ×, S V; ●, S VIII.

6. Discussion

In this section some problems related to the present theory are discussed. First, in our calculation we have dealt with one type of characteristic curve corresponding to the receding wave only. Here we show the justification for neglect of the other characteristics.

Using the variables defined by (2.7) and the expansions

$$u = a_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots,$$

$$a = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \ldots,$$

one can easily see that the characteristic curves $dx/dt = u + a$ become for the advancing characteristics C_+

$$\epsilon\, dx/d\tau = 2a_0 + \epsilon(u_1 + a_1) + \ldots$$

and for the receding characteristic C_-

$$\epsilon\, dx/d\tau = \epsilon(u_1 - a_1) + \ldots.$$

Thus C_- gives the equation obtained in §2, while C_+ has the gradient $2a_0/\epsilon$, which implies an almost instantaneous propagation of signals. This is reflected in the term $F(\tau)$ in (2.14) which means that the disturbances at the boundary propagate with an almost infinite velocity.

The method of determining the shock-propagation law given in §3 could be extended to the case of strong shock waves because (3.6) is valid also for strong shocks if we replace M_1 by M and a_0 by a, provided that the states in front of and behind the shock can be expanded about the respective constant states. Finally, it may be noted that the present theory is based upon the expansion about a constant state but the expansion about an arbitrarily steady state is possible (Asano 1970).

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